MTH 452,

Fall 2007 December 13

Final Exam – A – Solution

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Name ....

You have 3 hours and you have to answer 12 questions. Answer 10 out of questions 1–13, and you have to answer questions 14 and 15. Mark clearly which three questions are **not** to be graded. Each of the question 1–13 is worth 14 points and each of 14 and 15 is worth 10 points (total of 160). Show full logic for full credit. You may use one page written freely on two sides.
Good luck!

1. (a) Let $f(x) = x^3 - x$. What does Rolle's theorem guarantee? Explain and find it.

(b) If f and g are differentiable on (a, b) prove that between each two consecutive roots of f there is a root of $f' + fg'$. (Hint: use $h = f\exp(g)$.)

A: (a) We note that $f(0) = f(1) = 0$, the function is continuous on $[0, 1]$ and differentiable on $(0, 1)$, hence, by the Rolle's theorem there is $c \in (0, 1)$ such that $f'(c) = 0$. Clearly, $c = 1/\sqrt{3}$ since $f' = 3x^2 - 1$.

(b) Let $\alpha, \beta \in (a, b)$ be two consecutive zeros of f, that is, $f(\alpha) = f(\beta) = 0$ and f doesn't vanish on (α, β). Consider the function $h = f\exp(g)$, then $h(\alpha) = h(\beta) = 0$, and by Rolle's theorem there exists $c \in (\alpha, \beta)$ such that $h'(c) = 0$. Thus,

$$0 = h'(c) = f'(c)e^{g(c)} + f(c)g'(c)e^{g(c)}.$$

Dividing by $\exp(g(c)) \neq 0$ provides the desired result.

2. Let $f(x) = [x]$ be the largest integer function (i.e., $[x] = n$, where n is the largest integer $n < x$). At which points is the function F, given by

$$F(x) = \int_0^x (4 + [t])\, dt,$$

differentiable, and find its derivative there.

A: Let $n \in \mathbb{N}$ be fixed and let $n < x < n+1$, then $[x] = n$, and since the function $f(x) = 4 + [x] = 4 + n$ is continuous at x, it follows from the "Indefinite Riemann Integral Theorem" (Thm. 6.4.4) that

$$F'(x) = 4 + n, \qquad n < x < n + 1.$$

We conclude that F is differentiable at each $x \in (n, n+1)$. Next, we note that

$$\lim_{x \to n+0} F'(x) = 4 + n, \qquad \lim_{x \to n-0} F'(x) = 4 + (n-1) = 3 + n,$$

and since the limits are different, F' does not exist when x is integer.

3. Prove that if f is a continuous function that is defined on the interval $[a, b]$ and satisfies $f(x) > 0$ there, then there exists $\epsilon > 0$ such that $f(x) \geq \epsilon$ on $[a, b]$.

A: Since f is continuous on the closed interval $[a, b]$, it achieves there its maximum M and its minimum m. Now, $m > 0$ since there is a point $c \in [a, b]$ such that $f(c) = m > 0$.

Clearly, each $0 < \epsilon \leq m$ yields the desired conclusion.

4. Given that $f, g : [a, b] \to \mathbb{R}$ are such that $g(x) = f(x)$, except for one point $c \in (a, b)$ where $g(c) = f(c) + e$ $(e > 0)$, and f is Riemann integrable. Prove that $g \in R[a, b]$ and $\int_a^b g = \int_a^b f$.

A: Let $h(x) = g(x) - f(x)$, then $h(x) = 0$ for all $x \neq c \in [a, b]$, and $h(c) = e$. For each partition P_n of $[a, b]$ we have that $L(P_n, h) = 0$, since in each subinterval $[x_{k-1}, x_k]$, for $k = 1, \ldots n$, $m_k = \min h = 0$. If c belongs to the subinterval $[x_{l-1}, x_l]$, then $U(P, h) = e\Delta x_l$, since $M_k = \max h = 0$ for all $k \neq l$, and $M_l = e$. Now, as we refine the partition, i.e., as $n \to \infty$, then $\Delta x_l \to 0$ and, therefore, $\lim_{n\to\infty} U(P_n, h) = 0$. We conclude that

$$\int_a^b h = lim_{n\to\infty} L(P_n, h) = lim_{n\to\infty} U(P_n, h) = 0.$$

Therefore, h is Riemann integrable, $\int_a^b h = 0$, and hence so is $g = f + h$ as a sum of two Riemann integrable functions. Finally, since $\int_a^b h = 0$,

$$\int_a^b g = \int_a^b f.$$

A different proof is the following. Let $\epsilon >$ be small, then since f is Riemann integrable $f \leq M$ on $[a, b]$ and so $g \leq M + e$ on $[a, b]$ and letting $h = g - f$ we may write

$$\lim_{\epsilon\to 0} \int_a^b h = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} h + \lim_{\epsilon\to 0} \int_{c-\epsilon}^{c+\epsilon} h + \lim_{\epsilon\to 0} \int_{c+\epsilon}^{b} h.$$

Now, $h = 0$ on the intervals $[a, c - \epsilon]$ and $[c + \epsilon, b]$, and also $0 \leq h \leq e$, therefore

$$\lim_{\epsilon\to 0} \int_a^b h = \lim_{\epsilon\to 0} \int_{c-\epsilon}^{c+\epsilon} h \leq e \lim_{\epsilon\to 0} \int_{c-\epsilon}^{c+\epsilon} dx \leq e \lim_{\epsilon\to 0} 2\epsilon = 0.$$

Therefore, h is Riemann integrable, $\int_a^b h = 0$, and hence so is $g = f + h$ as a sum of two Riemann integrable functions. Finally, since $\int_a^b h = 0$, $\int_a^b g = \int_a^b f$.

5. Show that the equation $3^x = 5x$ has a solution in $(0, 1)$ using: (i) The Bolzano Intermediate Value Theorem; (ii) The Brouwer Fixed-point Theorem.

A: (i) Let $f(x) = 3^x - 5x$. Then, $f(0) = 1$ and $f(1) = -2$. The function f is continuous on $[0, 1]$ so by the The Bolzano Intermediate Value Theorem there is $c \in (0,1)$ such that $f(c) = 0$. This c is a solution of the equation. Since f is monotone decreasing in $(0,1)$ ($f' < 0$), c is the unique solution.

(ii) define the function $f(x) = 3^x/5$. Then $f : [0,1] \to [1/5,\, 3/5] \subseteq [0,1]$. Clearly the function is continuous, so by the Brouwer Fixed-point Theorem it has a fixed point $c \in (0,1)$ such that $f(c) = c$. We conclude, by the definition of f, that $3^c = 5c$.

6. State the Fundamental Theorem of Calculus (Do not prove it).

Use the theorem to find the value of

$$\int_0^1 (\cos t) e^{\sin t}\, dt.$$

A: The Fundamental Theorem of Calculus: Suppose that the function $f : [a,b] \to \mathbb{R}$ is differentiable, and $f' \in R[a,b]$. Then,

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

We note that if we define $f(x) = e^{\sin x}$, then

$$f'(x) = (\cos x) e^{\sin x}.$$

The function f is differentiable on $\mathbb{R}$ and f' is continuous and bounded on $[0,1]$, hence Riemann integrable. Thus, by the theorem

$$\int_0^1 (\cos t) e^{\sin t}\, dt = e^{\sin 1} - e^{\sin 0} = e^{\sin 1} - 1.$$

7. Assume that $f : (0,\infty) \to \mathbb{R}$ satisfies

$$\lim_{x\to\infty} (f(x) + f'(x)) = 0.$$

If $\lim_{x\to\infty} f(x) = L$, find L.

A: Since $\lim_{x\to\infty} f(x) = L$ exists and is finite, we have $\lim_{x\to\infty} f'(x) = -L$.

Given $\epsilon > 0$ there exists $M > 0$ such that for all $M < x, y$

$$|f(x) - f(y)| = |f(x) - L + L - f(y)| \leq |f(x) - L| + |L - f(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows from the Mean Value Theorem that for each pair x and y such that $M < x < y$, there exists $x < c < y$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Thus,

$$|f'(c)| \leq \frac{|f(y) - f(x)|}{y - x} \leq \frac{\epsilon}{y - x}.$$

We may choose $y - x$ to be large (it is enough if $y - x > 1$), so it follows that $|f'(c)| \leq \epsilon$, and ϵ is arbitrary, hence $|f'(c)| = 0 = |L|$.

We conclude that $\lim_{x\to\infty} f(x) = L = 0$.

Note: The function

$$f(x) = \frac{1}{x}\sin(x^2),$$

satisfies $\lim_{x\to\infty} f(x) = 0$, however,

$$f'(x) = -\frac{1}{x^2}\sin(x^2) + 2\cos(x^2),$$

and $\lim_{x\to\infty} f(x)$ DNE.

8. Determine whether the following functions are uniformly continuous. If f is uniformly continuous, explain it. If not, explain why not.

1. $f(x) = x^3$ with $x \in (-1, 1)$.
2. $f(x) = x^3$ with $x \in \mathbb{R}$.
3. $f(x) = e^{-3/x}$ with $x \in (0, 1)$.
4. $f(x) = e^{3/x}$ with $x \in (0, 1)$.

A:

1. Since the function $f(x) = x^3$ is continuous on $[-1, 1]$ and the interval is bounded and closed, it is uniformly continuous on $[-1, 1]$ by theorem and, therefore, also on $(-1, 1)$.

2. The function $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$. If it were, then by theorem it would have been bounded, but it is not.

Another proof. Let $\epsilon > 0$ and choose $x_n = n$ and $t_n = n + 1/n$. Then

$$|x_n - t_n| = \frac{1}{n},$$

and it can be made as small as we wish by choosing n sufficiently large. On the other hand,

$$|f(x_n) - f(t_n)| = |n^3 - (n + \frac{1}{n})^3| \geq 3n + 1.$$

Therefore, if $0 < \epsilon < n$ and $0 < \delta$ we have that

$$|f(x_n) - f(t_n)| > \epsilon,$$

although by choosing n large enough $|x_n - t_n| < \delta$.

3. We note that $\lim_{x\to 0} \exp(-3/x) = 0$, hence we may extend f to the closed interval $[0, 1]$ as a continuous function. Therefore, f is uniformly continuous on $[0, 1]$ and so on $(0, 1)$.

4. We note that $\lim_{x\to 0} \exp(3/x) = +\infty$, hence it is not bounded on $(0, 1)$ and cannot be extended to the closed interval $[0, 1]$ as a continuous function. Therefore, f is not uniformly continuous on $(0, 1)$.

9. If a statement is true, prove it. If not, provide a counterexample.

(i) If f is uniformly continuous on (a, b) then it is bounded on (a, b).

(ii) If f is bounded on (a, b) then it is uniformly continuous on (a, b).

A: (i) True. Since f is uniformly continuous on (a, b) it can be extended as a continuous function to the closed interval $[a, b]$, and so it is bounded.

(ii) False. Consider the function f given, for some $a < c < b$ by

$$f(x) = \begin{cases} 1 & \text{for } a \leq x \leq c, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is bounded on $[a, b]$ but not continuous, so not uniformly continuous.

10. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and at each $x \in \mathbb{R}$

$$f'(x) = f(x),$$

then $f(x) = ce^x$, for some $c \in \mathbb{R}$.

A: We rewrite the expression as

$$1 = \frac{f'(x)}{f(x)} = \frac{d}{dx} \ln(f(x)).$$

Integration yields $\ln(f(x)) = x + c$, where c is a constant. Therefore,

$$f(x) = ce^x.$$

11. State the Theorem on the two options that an increasing sequence $\{a_n\}$ has. (Do not prove it).

Use the theorem to find the limit of the sequence $\{a_n\}$, where

$$a_n = 1 - e^{-n/2}.$$

(You have to show that the sequence is increasing and ...)

A: For an increasing sequence $\{a_n\}$, there are two possibilities:
(a) $\{a_n\}$ is bounded above by M, and then $\lim_{n\to\infty} a_n = L \leq M$;
(b) $\{a_n\}$ is not bounded above, and then $\lim_{n\to\infty} a_n = \infty$.

We show that the sequence $a_n = 1 - e^{-n/2}$ is increasing and bounded. Since $0 < e^{-n/2} < 1$ for large n, we have that $a_n < 1$. Next, $e^{-(n+1)/2} < e^{-n/2}$ for all n, and so

$$a_{n+1} = 1 - e^{-(n+1)/2} > 1 - e^{-n/2} = a_n.$$

We conclude that the sequence is bounded above and increasing, and so the theorem guarantees that it converges to L. However, we note that $\lim_{n\to\infty} e^{-n/2} = 0$, hence $L = 1$.

12. Assume that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and define the function $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\, dt.$$

Prove that F is Lipschitz.

A: Since f is Riemann integrable on $[a,b]$, it is bounded, say $|f(x)| \leq M$ for $x \in [a,b]$. Let now $x, y \in [a,b]$ such that $x < y$. Then by the properties of the Riemann integral,

$$|F(y) - F(x)| = |\int_a^y f(t)\,dt - \int_a^x f(t)\,dt|$$

$$= |\int_x^y f(t)\,dt + \int_a^x f(t)\,dt - \int_a^x f(t)\,dt| = |\int_x^y f(t)\,dt| \leq \int_x^y |f(t)|\,dt$$

$$\leq M \int_x^y dt = M|y-x|.$$

Therefore, F is Lipschitz with Lipschitz constant L such that $L \leq M$.

13. (i) Assume that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence such that the set of values $S = \{a_n : 1 \leq n\}$ is finite. Prove that the sequence is constant from some $1 \leq n^*$ onward.

(ii) Prove that the sequence

$$a_n = \frac{n}{n+1}$$

is a Cauchy sequence. What is the limit?

A: (i) Let $S = \{s_1, s_2, \ldots, s_K\}$, so that there are K different values in the sequence. Next, let $d = \min_{i,j=1,\ldots,K}(|s_i - s_j|)$ be the smallest distance between any two values or numbers in S. Since$\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, given $\epsilon = d/2$ there exists $1 < n^*$ such that

$$|a_m - a_n| \leq \frac{1}{2}d, \qquad \text{for all } \; n^* < n, m.$$

Now, fix m and note that $a_m \in S$, therefore $a_m = s_{j*}$ for some $j* \in \{1, 2, \ldots, K\}$. If for one $n^* < n$ we have that $a_n \neq s_{j*}$, then there is $j^{**} \in \{1, 2, \ldots, K\}$ and $a_n = s_{j^{**}}$, while $s_{j^{**}} \neq s_{j*}$. But, this contradicts the fact that $|a_m - a_n| \leq \frac{1}{2}d$ while $|s_{j**} - s_{j*}| \geq d$, indeed, it follows from the definition of d that

$$d \leq |s_{j**} - s_{j*}| = |a_m - a_n| \leq \frac{1}{2}d,$$

which is impossible. We conclude that if $n^* < m$ and $a_m = s_{j*}$ for some $j*$, then $a_n = s_{j*}$ for all $n^* < n$. Therefore, starting from $n^* + 1$ the sequence is just constant.

(ii) Let $\epsilon > 0$ be given. We assume that $n < m$ and compute the difference

$$|a_m - a_n| = \left|\frac{m}{m+1} - \frac{n}{n+1}\right| = \frac{m-n}{mn} \leq \frac{m}{mn} \leq \frac{1}{n}.$$

We now choose $1/\epsilon < n^*$ and obtain that $|a_m - a_n| \leq \epsilon$ for all $n^* < n, m$. We conclude that the sequence is Cauchy. The limit is

$$\lim_{n\to\infty} a_n = \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

14. (You have to answer this question.) If a statement is true, prove it. If not, provide a counterexample.

(i) If f and g are differentiable in (a, b) and $f(x)g(x) = 1$ for each $x \in (a, b)$, then

$$\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} = 0.$$

(ii) If f is increasing on (a, b) then $f' \geq 0$ at each point in (a, b).

A: (i) True. Clearly, $f(x) \neq 0$ and $g(x) \neq 0$ for each $x \in (a, b)$, otherwise $f(x)g(x) = 0$ somewhere. Next, since both functions are differentiable, we have

$$0 = \frac{d}{dx}(1) = \frac{d}{dx}(fg) = f'g + fg',$$

where we used the product rule. Dividing now by fg we obtain the assertion.

(ii) False. The function

$$f(x) = \begin{cases} \dfrac{x-a}{c-a} & \text{for } \ a \leq x \leq c, \\ 2(x-c)+1 & \text{for } \ c \leq x \leq b, \end{cases}$$

for $c \in (a, b)$, is continuous and increasing but is not differentiable at $x = c$ since $f'(c-) = 1$ while $f'(c+) = 2$.

15. (You have to answer this question.) Prove that if $f : [a, b] \to [a, \infty)$ is Lipschitz, with Lipschitz constant $L < 1$, then f has a unique fixed point in $[a, \infty)$.

A: If $f(a) = a$, we are done. So assume that $f(a) > a$ and we need to show that there is $a < b$ such that $f(b) \leq b$, and then Brouwer's fixed-point theorem is applicable (Why?). Now, since f is Lipschitz with $L < 1$, we have for each $x \in [a, \infty)$

$$|f(x) - f(a)| < L(x - a),$$

and thus, $|f(x)| \leq |f(a)| + L(x - a)$. Let $g(x) = |f(x)| - x$, then, $g(a) = |f(a)| - a > 0$ and

$$g(x) \leq |f(a)| + L(x - a) - x = |f(a)| - La - (1 - L)x.$$

Since $L < 1$ for large x it follows that $g(x) < 0$ so $|f(b)| \leq b$ for some b, and using now Brouwer's fixed-point theorem yields the desired result.