

APM-541 Fall 2004
SOLUTION of EXAM 1-A

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You have 105 minutes. Answer 5 questions out of 1–8, and answer questions 9 and 10. Mark clearly which questions are **not** to be graded. Each question 1–8 is worth 20 points and questions 9 and 10 are worth 10 points each (total of 120). You may use a photocopy of the Laplace transforms and a two pages freely written. Please attach the pages to the exam. **Show full logic for full credit.**

Good luck!

1. Given $F(s)$ find $f(t)$.

$$F(s) = \frac{1}{s(s-a)(s-b)}, \quad a \neq b.$$

A. It follows from (11) in Table 5.9 that

$$\mathcal{L}^{-1} \left(\frac{1}{(s-a)(s-b)} \right) = \frac{1}{(a-b)} (e^{at} - e^{bt}).$$

Therefore, by using the tables we find

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \left(\frac{1}{s(s-a)(s-b)} \right) = \frac{1}{(a-b)} \int_0^t (e^{a\tau} - e^{b\tau}) d\tau \\ &= \frac{1}{(a-b)} \left(\frac{1}{a} (e^{at} - 1) - \frac{1}{b} (e^{bt} - 1) \right). \end{aligned}$$

2. Find the rank of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

A. We add the third row to the first row and then divide it by 2; we multiply the third row by 4 and add to the second row and divide it by 3; we move the third row to be the first row, then we subtract the second from the third, and we obtain

$$\begin{pmatrix} 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \\ -1 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This means that two rows are linearly independent, so

$$\text{rank}(A) = 2.$$

3. Solve the initial value problem (a is a positive number)

$$y'' + 4y = r(t), \quad y(0) = y'(0) = 0, \quad r(t) = t \text{ if } 0 < t < a, \text{ and } r(t) = 0 \text{ otherwise.}$$

A. First we rewrite r as $r(t) = t(1 - u(t - a))$, and then

$$r(t) = t - (t - a + a)u(t - a) = t - (t - a)u(t - a) - au(t - a).$$

Then, let $Y = \mathcal{L}(y)$ and apply the Laplace transform to the equation, thus,

$$Y = \frac{1}{s^2(s^2 + 4)} - \frac{e^{-as}}{s^2(s^2 + 4)} - a \frac{e^{-as}}{s(s^2 + 4)}.$$

Next, we find in Table 5.9 that the inverses are

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2 + 4)}\right) = \frac{1}{8}(2t - \sin 2t), \quad \mathcal{L}^{-1}\left(\frac{1}{s(s^2 + 4)}\right) = \frac{1}{4}(1 - \cos 2t),$$

Finally, using these in the above yields the solution

$$y(t) = \frac{1}{8}(2t - \sin 2t) - \frac{1}{8}(2(t - a) - \sin 2(t - a))u(t - a) - \frac{a}{4}(1 - \cos 2(t - a))u(t - a).$$

4. How does the rank of A depends on a and b ?

$$\mathbf{A} = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}.$$

A. We need to bring A to an upper triangular form. We subtract the first row from the second and obtain

$$\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & b - a \end{pmatrix}.$$

We conclude that if $b \neq a$ then the two rows are linearly independent and $\text{rank}(A)=2$. If $a = b$ then the second row has only zeros and thus $\text{rank}(A)=1$.

5. Use Gauss elimination to find **all** the solutions of the system

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ -3x + 3y - 6z + v &= 3 \\ x - y + 2z &= 0 \end{aligned}$$

A. The augmented matrix is

$$\tilde{A} = \begin{pmatrix} 2 & -2 & 4 & 0 & 0 \\ -3 & 3 & -6 & 1 & 3 \\ 1 & -1 & 2 & 0 & 0 \end{pmatrix}.$$

We divide the first row by 2 and use it in the other two rows, thus

$$\tilde{A} = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 \\ -3 & 3 & -6 & 1 & 3 \\ 1 & -1 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the rank of A is 2 and two of the variables, say y, z , are parameters. The solution is

$$x = y - 2z, \quad v = 3.$$

6. Solve the initial value problem (b is a constant)

$$y'' - 16y = b\delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 2.$$

A. First, we note that $\mathcal{L}(\delta(t - \pi)) = e^{-\pi s}$. Taking the Laplace transform, and denoting $Y = \mathcal{L}(y)$ yields

$$Y = \frac{2}{s^2 - 16} + e^{-\pi s} \frac{b}{s^2 - 16}.$$

Using the tables now leads to

$$y(t) = \frac{1}{2} \sinh 4t + \frac{b}{4} \sinh 4(t - \pi) u(t - \pi).$$

7. Find the inverse transform of $F(s)$, where b and d are positive constants,

$$F(s) = \ln \left(\frac{s^2 - d^2}{(s - b)^2} \right).$$

A. We rewrite it as

$$F(s) = \ln(s^2 - d^2) - 2 \ln(s - b).$$

Then,

$$-F' = -\frac{2s}{s^2 - d^2} + \frac{2}{s - b}.$$

Now, $-F' = \mathcal{L}(tf)$ and from the tables we find

$$tf(t) = -2 \cosh dt + 2e^{bt}.$$

Therefore,

$$f(t) = \frac{-2}{t} \cosh dt + \frac{2}{t} e^{bt}.$$

8. Use the Laplace transform to solve the integral equation for y ,

$$y(t) = te^t - 2e^t \int_0^t e^{-\tau} y(\tau) d\tau.$$

A. We rewrite the equation as

$$y(t) = te^t - 2 \int_0^t e^{t-\tau} y(\tau) d\tau$$

and, thus, $y = te^t - 2e^t * y$. Its Laplace transform is

$$Y = \frac{1}{(s-1)^2} - Y \frac{2}{(s-1)}.$$

Hence,

$$(s+1)Y = \frac{1}{s-1} \implies Y = \frac{1}{s^2-1}.$$

The solution is $y(t) = \sinh t$.

9. [10pts](You have to answer this question!) Find the Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ \sin t & \text{if } 2\pi < t \end{cases}.$$

A. We rewrite f by using the u function as

$$\begin{aligned} f(t) &= t(1 - u(t - \pi)) + \sin t u(t - 2\pi) \\ &= t - (t - \pi + \pi)u(t - \pi) + \sin(t - 2\pi)u(t - 2\pi) \\ &= t - (t - \pi)u(t - \pi) - \pi u(t - \pi) + \sin(t - 2\pi)u(t - 2\pi). \end{aligned}$$

Here, we used the 2π periodicity of $\sin t$.

Then, using the tables we obtain that $\mathcal{L}(f) = F(s)$ is given by

$$F(s) = \frac{1}{s^2} - e^{-\pi s} \left(\frac{1}{s^2} + \frac{\pi}{s} \right) + e^{-2\pi s} \frac{1}{s^2 + 1}.$$

10. [10pts](You have to answer this question!) For which values of d is the set of all vectors of the form

$$\mathbf{v} = (v_1, v_2, v_3, v_4, v_5, v_6),$$

where $v_1 = d$, $v_2 = v_3 = 0$, a vector space? When it is a vector space find its dimension and a basis.

A. First, we note that the set is the collection V of all the vectors of the form

$$\mathbf{v} = (d, 0, 0, v_4, v_5, v_6).$$

If $d \neq 0$ then the vector $\mathbf{v} = (d, 0, 0, 0, 0, 0)$ is in the set but the vector $2\mathbf{v} = (2d, 0, 0, 0, 0, 0)$ is not in the set and, therefore, V is NOT a vector space since it is not closed under multiplication by a real number.

For $d = 0$ the collection V is a vector space. A possible basis for V is

$$\mathbf{i} = (0, 0, 0, 1, 0, 0), \quad \mathbf{j} = (0, 0, 0, 0, 1, 0), \quad \mathbf{k} = (0, 0, 0, 0, 0, 1),$$

and it is straightforward to see that each vector in this vector space can be written as a linear combination of the basis vectors. Moreover its dimension of V is 3.

Thus, V is a three-dimensional subspace of the 6-dimensional space \mathbb{R}^6 .