APM-541 Fall 2004
SOLUTION of EXAM 1-B

M. Shillor

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You have 105 minutes. Answer 5 questions out of 1–8, and answer questions 9 and 10. Mark clearly which questions are **not** to be graded. Each question 1–8 is worth 20 points and questions 9 and 10 are worth 10 points each (total of 120). You may use a photocopy of the Lapace transforms and a two pages freely written. Please attach the pages to the exam. **Show full logic for full credit.**

Good luck!

1. Given $F(s)$ find $f(t)$.

$$F(s) = \frac{1}{s(s-d)(s-b)}, \qquad d \neq b.$$

A. It follows from (11) in Table 5.9 that

$$\mathcal{L}^{-1}\left(\frac{1}{(s-d)(s-b)}\right) = \frac{1}{(d-b)}(e^{dt} - e^{bt}).$$

Therefore, by using the tables we find

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s(s-d)(s-b)}\right) = \frac{1}{(d-b)}\int_0^t (e^{d\tau} - e^{b\tau})\,d\tau$$

$$= \frac{1}{(d-b)}\left(\frac{1}{d}(e^{dt}-1) - \frac{1}{b}(e^{bt}-1)\right) = \frac{1}{bd} + \frac{e^{dt}}{d(d-b)} - \frac{e^{bt}}{b(d-b)}.$$

2. Find the rank of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

A. We add the third row to the first row and then divide it by 2; we multiply the third row by 4 and add to the second row and divide it by 3; we move the third row to be the first row, then we subtract the second from the third, and we obtain

$$\begin{pmatrix} 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \\ -1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This means that two rows are linearly independent, so

$$\text{rank } (A) = 2.$$

3. Solve the initial value problem (a is a positive number)

$$y'' + 9y = r(t), \quad y(0) = y'(0) = 0, \quad r(t) = t \text{ if } 0 < t < a, \text{ and } r(t) = 0 \text{ else.}$$

A. First we rewrite r as $r(t) = t(1 - u(t-a))$, and then

$$r(t) = t - (t - a + a)u(t-a) = t - (t-a)u(t-a) - au(t-a).$$

Then, let $Y = \mathcal{L}(y)$ and apply the Laplace transform to the equation, thus,

$$Y = \frac{1}{s^2(s^2+9)} - \frac{e^{-as}}{s^2(s^2+9)} - a\frac{e^{-as}}{s(s^2+9)}.$$

Next, we find in Table 5.9 that the inverses are

$$\mathcal{L}^{-1}\Big(\frac{1}{s^2(s^2+9)}\Big) = \frac{1}{8}(2t - \sin 3t), \qquad \mathcal{L}^{-1}\Big(\frac{1}{s(s^2+9)}\Big) = \frac{1}{4}(1 - \cos 3t),$$

Finally, using these in the above yields the solution

$$y(t) = \frac{1}{27}(3t - \sin 3t) - \frac{1}{27}(3(t-a) - \sin 3(t-a))u(t-a) - \frac{a}{9}(1 - \cos 3(t-a))u(t-a).$$

4. How does the rank of A depends on a and b?

$$\mathbf{A} = \begin{pmatrix} 1 & b \\ 1 & a \end{pmatrix}.$$

A. We need to bring A to an upper triangular form. We subtract the first row from the second and obtain

$$\mathbf{A} = \begin{pmatrix} 1 & b \\ 0 & a-b \end{pmatrix}.$$

We conclude that if $b \neq a$ then the two rows are linearly independent and rank(A)=2. If $a = b$ then the second row has only zeros and thus rank(A)=1.

5. Use Gauss elimination to find **all** the solutions of the system

$$\begin{aligned} 2x - 2y + 4z &= 0 \\ -3x + 3y - 6z + 2v &= 6 \\ x - y + 2z &= 0 \end{aligned}$$

A. The augmented matrix is

$$\tilde{A} = \begin{pmatrix} 2 & -2 & 4 & 0 & 0 \\ -3 & 3 & -6 & 2 & 6 \\ 1 & -1 & 2 & 0 & 0 \end{pmatrix}.$$

We divide the first row by 2 and use it in the other two rows, thus

$$\tilde{A} = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 \\ -3 & 3 & -6 & 2 & 6 \\ 1 & -1 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the rank of A is 2 and two of the variables, say y, z, are parameters. The solution is

$$x = y - 2z, \qquad v = 3.$$

6. Solve the initial value problem (b is a constant)

$$y'' - 9y = b\delta(t - \pi), \qquad y(0) = 0, \quad y'(0) = 4.$$

A. First, we note that $\mathcal{L}(\delta(t - \pi)) = e^{-\pi s}$. Taking the Laplace transform, and denoting $Y = \mathcal{L}(y)$ yields

$$Y = \frac{4}{s^2 - 9} + e^{-\pi s}\frac{b}{s^2 - 9}.$$

Using the tables now leads to

$$y(t) = \frac{4}{3}\sinh 3t + \frac{b}{3}\sinh 3(t - \pi)\, u(t - \pi).$$

7. Find the inverse transform of $F(s)$, where b and d are positive constants,

$$F(s) = \ln\left(\frac{s^2 - b^2}{(s - d)^2}\right).$$

A. We rewrite it as

$$F(s) = \ln(s^2 - b^2) - 2\ln(s - d).$$

Then,

$$-F' = -\frac{2s}{s^2 - b^2} + \frac{2}{s - d}.$$

Now, $-F' = \mathcal{L}(tf)$ and from the tables we find

$$tf(t) = -2\cosh bt + 2e^{dt}.$$

Therefore,

$$f(t) = \frac{-2}{t}\cosh bt + \frac{2}{t}e^{dt}.$$

8. Use the Laplace transform to solve the integral equation for y,

$$y(t) = te^t - 3e^t \int_0^t e^{-\tau} y(\tau)\, d\tau.$$

A. We rewrite the equation as

$$y(t) = te^t - 3 \int_0^t e^{t-\tau} y(\tau)\, d\tau$$

and, thus, $y = te^t - 3e^t * y$. Its Laplace transform is

$$Y = \frac{1}{(s-1)^2} - Y\frac{3}{(s-1)}.$$

Hence,

$$(s+2)Y = \frac{1}{s-1} \quad \Longrightarrow \quad Y = \frac{1}{(s-1)(s+2)} = \frac{1}{3(s-1)} - \frac{1}{3(s+2)}.$$

The solution is $y(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$.

9. [10pts](You have to answer this question!) Find the Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ \cos t & \text{if } 2\pi < t \end{cases}.$$

A. We rewrite f by using the u function as

$$\begin{aligned} f(t) &= t(1 - u(t-\pi)) + \cos t\, u(t-2\pi) \\ &= t - (t - \pi + \pi)u(t-\pi) + \cos(t-2\pi)u(t-2\pi) \\ &= t - (t-\pi)u(t-\pi) - \pi u(t-\pi) + \cos(t-2\pi)u(t-2\pi). \end{aligned}$$

Here, we used the 2π periodicity of $\cos t$.

Then, using the tables we obtain that $\mathcal{L}(f) = F(s)$ is given by

$$F(s) = \frac{1}{s^2} - e^{-\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right) + e^{-2\pi s}\frac{s}{s^2+1}.$$

10. [10pts](You have to answer this question!) For which values of d is the set of all vectors of the form

$$\mathbf{v} = (v_1, v_2, v_3, v_4, v_5),$$

where $v_1 = m$, $v_2 = v_3 = 0$, a vector space? When it is a vector space find its dimension and a basis.

A. First, we note that the set is the collection V of all the vectors of the form

$$\mathbf{v} = (d, 0, 0, v_4, v_5, v_6).$$

If $m \neq 0$ then the vector $\mathbf{v} = (d, 0, 0, 0, 0)$ is in the set but the vector $2\mathbf{v} = (2m, 0, 0, 0, 0)$ in not in the set and, therefore, V is NOT a vector space since it is not closed under multiplication by a real number.

For $m = 0$ the collection V is a vector space. A possible basis for V is

$$\mathbf{i} = (0, 0, 0, 1, 0), \qquad \mathbf{j} = (0, 0, 0, 0, 1),$$

and it is straightforward to see that each vector in this vector space can be written as a linear combination of the basis vectors. Moreover its dimension of V is 2.

Thus, V is a two-dimensional subspace of the 5-dimensional space $\mathbb{R}^5$.